2025

(Nov–Dec)

FYUP : 5th Semester Examination

MAJOR

MATHEMATICS

(Number Theory and Ring Theory)

MTH–301

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **four** questions, taking **one** from each Unit

UNIT—I

1. *(a)* State whether the following statements are true or false with brief justification (any *four*) (a, b, c, d denote positive integers) : 2½×4=10

(i) If p is a prime and $a^2 \equiv b^2 \pmod{p}$, then $p \mid a-b$.

(ii) If a is odd, then $(a+3,\ a-3)=6$.

(iii) $\exists a$ such that $5a^2+22$ is divisible by 17.

(iv) If $\underline{|p-1} \equiv -1 \pmod{p}$, then p is a prime.

(v) If $a^2 \mid c^3$, then $a \mid c$.

(vi) If $b \mid a^2+1$, then $b \mid a^4+1$.

(vii) The set of integers

$$\{1,\ 2,\ 4,\ 8,\ 7,\ -2,\ -4,\ -8\}$$

forms a reduced residue system modulo 15.

(b) State and prove Wilson's theorem. 5

(c) Find the last two digits in ordinary decimal representation of 7^{200}. 3

2. (a) Prove that $n^{12}-a^{12}$ is divisible by 91 if n and a are prime to 91. 3

(b) Show that 3, 6, 9, ..., $3m$ is a complete residue system mod m, if $3 \times m$. 3

(c) If q is a prime factor of a^2+b^2 and if $q \equiv 3 \pmod{4}$, show that $q \mid a$ and $q \mid b$. 4

(d) If p is an odd prime, show that

$$1^2 \cdot 3^2 \cdot 5^2 \ldots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}$$ 4

(e) Prove that $11 \mid 2^{4n+3}+5^{n+2}$ for every positive integer n. 4

UNIT—II

3. (a) Find all integers that give remainders 1, 2, 3 when divided by 4, 5 and 7 respectively. 5

(b) Solve the congruence $15x \equiv 25 \pmod{35}$. 4

(c) Show that $\mu(n)$ is multiplicative and

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$ 4

(d) If k is the number of distinct prime factors of n (a positive integer), show that

$$\sum_{d \mid n} |\mu(d)| = 2^k$$ 4

(e) Give an example to show that if $f(n)$ is totally multiplicative, $F(n)$ need not be totally multiplicative, when

$$F(n) = \sum_{d \mid n} f(d)$$ 2

4. *(a)* What is the highest power of 6 dividing $\underline{|523}$? 3

(b) If $\underline{|96}$ were written out in the ordinary decimal notation without factorial sign, find the number of zeros that would be at the right end. 3

(c) For any two real numbers x and y, show that

$$[x]+[y] \le [x+y] \le [x]+[y]+1$$ 4

(d) Prove that

$$\prod_{d \mid n} d = n^{\frac{d(n)}{2}}$$ 3

(e) If a and b are positive integers such that $(a, b)=1$ and c is a real number such that ac, bc are integers, prove that c is an integer. Hence show that

$$c = \frac{\underline{|n}}{\underline{|a}\,\underline{|b}}$$

is an integer if $(a, b)=1$ and $a+b=n+1$. 6

UNIT—III

5. *(a)* Let $a \in R$, a ring. Show that $\{x \in R : ax = 0\}$ is a subring of R. 4

(b) Show that a finite integral domain is a field. 5

(c) Let A be an ideal of a commutative ring $R \neq \{0\}$. Show that R/A is an integral domain if and only if A is a prime ideal. 5

(d) Let $J = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that J is a commutative ring under addition and multiplication of real numbers. Is it a field? Justify. 5

6. *(a)* If U and V are two ideals of R, show that

$$U+V = \{x+y : x \in U \text{ and } y \in V\}$$

is an ideal of R. 3

(b) In a ring R, if $x^2 = x \ \forall\ x \in R$, show that R is commutative. 5

(c) Show that the units of the ring

$$\mathbb{Z}[i] = \{a+ib : a, b \in \mathbb{Z}\}$$

are $+1$ and -1. 4

(d) Let R be a commutative ring with unity whose only ideals are (0) and R itself. Show that R is a field. 5

(e) In the ring of integers $\mathbb{Z}$, find a positive integer m s.t. $2\mathbb{Z}+3\mathbb{Z}=m\mathbb{Z}$. 2

UNIT—IV

7. *(a)* Define a ring homomorphism. Show that the map $\phi : \mathbb{Z}_4 \to \mathbb{Z}_{10}$ defined by $\phi(x) = 5x$ is a ring homomorphism. 1+3=4 *(b)* If $\phi: R \to S$ is a ring homomorphism, show that—

(i) $\phi(A)$ is an ideal of S whenever A is an ideal of R;

(ii) $\ker \phi$ is an ideal of R;

(iii) $\phi^{-1}(B)$ is an ideal of R for every ideal B of S;

(iv) R is commutative implies $\phi(R)$ is commutative. 2×4=8

(c) Show that the only ring automorphism of the set of rational numbers is the identity map. 4

(d) Show that the map $\phi: R \to \mathbb{Z}$ defined by

$$\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} \mapsto a$$

where

$$R = \left\{ \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}$$

is a ring homomorphism. 3

8. *(a)* Show that

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$$

is an integral domain but not a unique factorization domain. 5

(b) In a PID, show that an element is irreducible if and only if it is prime. 4

(c) In the integral domain $\mathbb{Z}[\sqrt{-3}]$, give an example of an irreducible element which is not a prime. 3

(d) Give an example of a prime ideal which is not maximal. 3

(e) In a finite commutative ring, show that every prime ideal is maximal. 4

★★★